Lec 7: 09/10/2014

Fermi Acceleration (Cont'd):

We now turn to an alternative geometry in which the scattering of test particles occurs within a converging flow, arising across a shock. As we will see, the second-order process that we discussed last time will turn into a first-order one in this case.

First, let us discuss the shock waves in some detail.

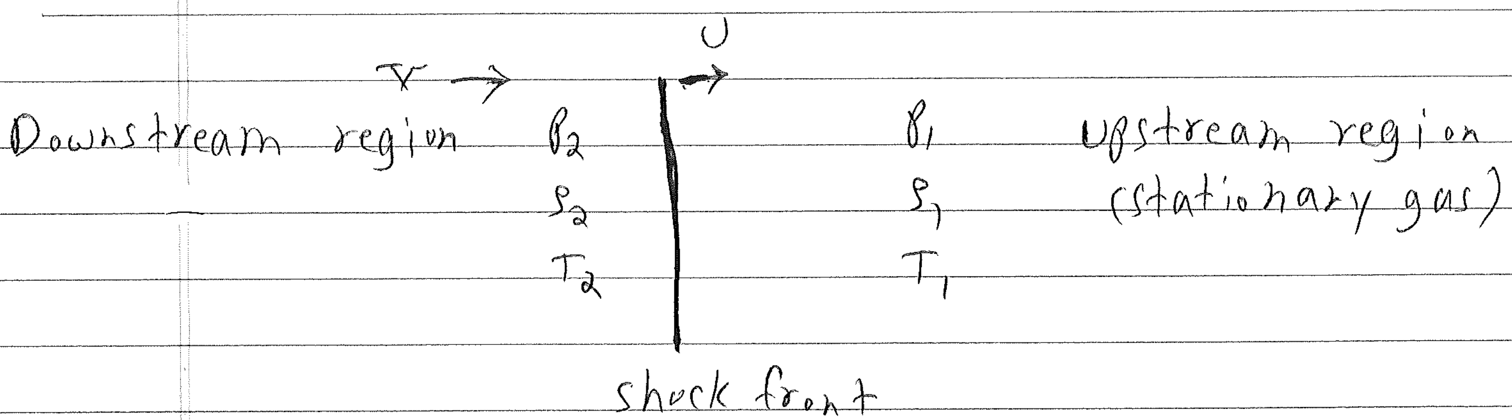
Shock waves are found ubiquitously in high energy astrophysics, and play a key role in many different astrophysical environments. It is a general property of perturbations in a gas that they are propagated away from their source at the speed of sound in the medium, c_s, given by:

$$c_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{ad}} = \sqrt{\frac{\gamma p}{\rho}} \qquad \left(\gamma \equiv \frac{c_p}{c_v} = \frac{c_v + 1}{c_v}\right)$$

Here "ad" refers to the adiabatic condition.

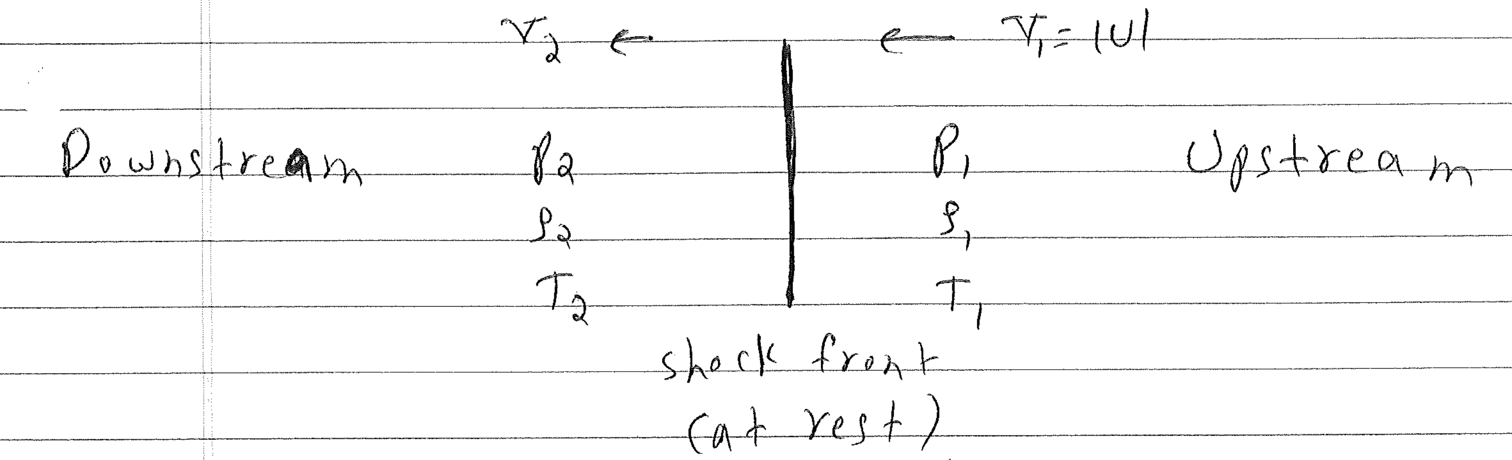
If the source and the gas have a relative velocity that is greater than c_s, then the disturbance cannot behave like a sound wave at all. There will be a discontinuity between the regions behind and ahead of the disturbance. These discontinuities are called shock waves.

Let us focus on a plane shock wave and assume abrupt discontinuity between the two regions of fluid flow:



It is convenient to transform to a reference frame moving at velocity U, in which the shock front is at stationary.

In this frame, the upstream region and the downstream region move at respective velocities v_1, v_2:



The behavior of the gas on passing through the shock front is described by a set of conservation relations. First, mass is conserved on passing through the discontinuity:

$$\rho_1 v_1 = \rho_2 v_2$$

Second, the energy flux is continuous. The energy flux through the shock front is $\rho_1 v_1 \left(\frac{1}{2} v_1^2 + \omega_1\right)$ and $\rho_2 v_2 \left(\frac{1}{2} v_2^2 + \omega_2\right)$ in the upstream and downstream regions respectively. Here, $\omega = PV + \epsilon$ is the enthalpy per unit mass, where ϵ is the internal energy per unit mass and $V = \rho^{-1}$ is the specific volume. Note that the $\frac{1}{2} v_1^2$ term is the kinetic energy per unit mass that arises because of the net flow of the gas. Conservation of energy flux therefore implies:

$$\rho_1 v_1 \left(\frac{1}{2} v_1^2 + \omega_1\right) = \rho_2 v_2 \left(\frac{1}{2} v_2^2 + \omega_2\right)$$

Finally, the momentum flux through the shock front must be continuous, which results in:

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$$

We note that, as expected, the momentum flux is conserved if pressure is the same in the upstream and downstream regions $P_1 = P_2$.

For an ideal gas, we have $\omega = \frac{\gamma P V}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \frac{P}{\rho}$. The mass flux is denoted by $j = \rho_1 v_1 = \rho_2 v_2$. We then find:

$$j^2 = \frac{P_2 - P_1}{V_1 - V_2} \qquad (V_1 = \rho_1^{-1},\ V_2 = \rho_2^{-1})$$

In addition:

$$v_1 - v_2 = j\,(V_1 - V_2) = \left[(P_2 - P_1)(V_1 - V_2)\right]^{\frac{1}{2}}$$

From the conservation of energy flux we obtain:

$$(\omega_1 - \omega_2) + \frac{1}{2}(V_1 + V_2)(P_2 - P_1) = 0$$

Using the relation $\omega = \frac{\gamma P V}{\gamma - 1}$ for an ideal gas, this leads to:

$$\frac{v_2}{v_1} = \frac{P_1(\gamma+1) + P_2(\gamma-1)}{P_1(\gamma-1) + P_2(\gamma+1)}$$

One can show that:

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}$$

Here, M_1 is the Mach number in the upstream region:

$$M_1 = \frac{v_1}{c_{s,1}}, \qquad c_{s,1} = \sqrt{\frac{\gamma P_1}{\rho_1}}$$

We then find:

$$\frac{v_1}{v_2} = \frac{\gamma+1}{(\gamma-1) + \frac{2}{M_1^2}}$$

For very strong shocks, $v_1 \gg c_{s,1}$, this results in:

$$\frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}$$

For an ideal monatomic gas, we have $\gamma = \frac{5}{3}$, and hence:

$v_1 = 4 v_2 \Rightarrow v_2 = \frac{|U|}{4}$

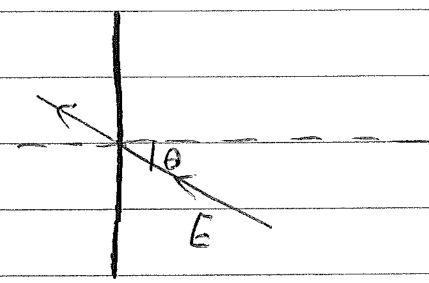
In the rest frame of the upstream region, gas in the downstream region moves at a speed $\frac{3U}{4}$ behind the shock front.

We now discuss the actual process of particle acceleration. A particle crossing from the upstream to dowstream sides of the shock acquires an increase in its energy according to:

$E' = \gamma\left(E + \frac{3}{4} U p_x\right)$

$\gamma = \left[1 - \left(\frac{3U}{4c}\right)^2\right]^{-\frac{1}{2}} \simeq 1$

$p_x = \frac{E}{c}\cos\theta$



shock front

Here, we have assumed that the particle that crosses the shock fround is relativistic, $E = \frac{p}{c}$, and the shock is non-relativistic $U \ll c$.

The energy increase is:

$$\Delta E = E' - E \Rightarrow \frac{\Delta E}{E} \simeq \frac{3}{4}\frac{U}{c}\cos\theta$$

Integrating over the incident angle $0 \leq \theta \leq \frac{\pi}{2}$, we find:

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{3}{4}\frac{U}{c}\int_0^{\frac{\pi}{2}} P_{(\theta)} \cos\theta \, d\theta \qquad P_{(\theta)} = 2\sin\theta\cos\theta$$

Thus:

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{1}{2}\frac{U}{c}$$

After crossing the shock front, the particle's velocity vector is randomized through elastic scatterings off the gas particles in the downstream region. The particle gains another fractional increase of $\frac{1}{2}\frac{U}{c}$ as it crosses the shock front back to the upstream region. This is the main difference with the simple one-dimensional example we considered earlier, in which the linear terms in head-on and catch-up collisions had opposite signs and cancelled out.

The average energy increase per round trip is:

$$\beta = 1 + \frac{1}{2}\frac{U}{c} + \frac{1}{2}\frac{U}{c} = 1 + \frac{U}{c}$$

Now, we need to work out the probability P that particle crosses the shock front back to the upstream region. According to kinetic theory, the flux of particles crossing a surface is $\frac{1}{4}nc$, where n is the number density of particles and particles are assumed to be relativistic. The flux represents the number of particles within one mean free path from the surface crossing it at incident angles $0 \leq \theta \leq \frac{\pi}{2}$.

In the case of the shock, however, the front moves at a velocity $\frac{1}{4}U$ relative to the downstream region. As a result, the flux to the downstream region is $\frac{1}{4}n(c + \frac{1}{2}U)$, while that from the downstream region is $\frac{1}{4}n(c - \frac{1}{2}U)$. This implies that the probability P for the particles to cross the shock front back is going to be:

$\rho = 1 - \frac{u}{c}$

Having found ρ and β, we can now derive the dependence of the spectrum on E. Starting with an initial energy E_0 and number N_0 in the upstream region, after k collisions we have:

$E = E_0 \beta^k$, $N = N_0 \rho^k$

Thus:

$\ln\left(\frac{E}{E_0}\right) = k \ln\beta$, $\ln\left(\frac{N}{N_0}\right) = k \ln \rho$

This results in:

$\frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{\frac{\ln\rho}{\ln\beta}}$ since $\frac{u}{c} \ll 1$ (non-relativistic limit)

$\frac{\ln\rho}{\ln\beta} = \frac{\ln(1-\frac{u}{c})}{\ln(1+\frac{u}{c})} \approx \frac{-\frac{u}{c}}{\frac{u}{c}} = -1$

Therefore:

$\frac{N}{N_0} \approx \left(\frac{E}{E_0}\right)^{-1} \Rightarrow \frac{dN}{dE} = N(E) \propto E^{-2}$

The value of the exponent is remarkably close to the "observed" universal value of -2.5.